Tail Invariant Measures of the Dyck Shift

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Abstract

We show that the one-sided Dyck shift has a unique tail invariant topologically σ-finite measure (up to scaling). This invariant measure of the one sided Dyck turns out to be a shift-invariant probability. Furthermore, it is one of the two ergodic probabilities obtaining maximal entropy. For the two sided Dyck shift we show that there are exactly three ergodic double-tail invariant probabilities. We show that the two sided Dyck has a double-tail invariant probability, which is also shift invariant, with entropy strictly less than the topological entropy.[1]

1 Introduction

The study of tail invariant probabilities for subshifts has so far focused mostly on sofic systems. There are known results for the case of the one sided tail of (mixing) SFT's [3]. Also, for the case of the β-shift it is known that there exists a unique tail-invariant measure [1]. In all of these examples the tail-invariant measure is also equivalent to a unique shift invariant measure of maximal entropy. Invariant measures for the double-tail (and some sub-relations of the double-tail) of SFT's have also been characterized [10].

Let Σ be a finite alphabet. For a subshift $X \subset \Sigma^{\mathbb{Z}}$, we define the double-tail relation, or *homoclinic* [10] relation of X to be:

$$\mathcal{T}_2(X) := \{(x, x') \in X \times X \ \ \exists n \geq 0 \ \forall |k| > n \ x_k = x'_k\}$$

A $\mathcal{T}_2(X)$-holonomy is an injective Borel function $g : A \mapsto g(A)$, with A a Borel set and $(x, g(x)) \in \mathcal{T}_2(X)$ for every $x \in A$. We say that $\mu \in \mathcal{M}(X)$ is a double-tail invariant if $\mu(A) = \mu(g(A))$ for every $\mathcal{T}_2(X)$-holonomy g.

In this paper we identify the tail invariant probability measures for the Dyck Shift. This subshift was used in [11] as a counter-example for a conjecture of B. Weiss, showing there are exactly two measures of maximal entropy for this subshift, both of which are Bernoulli. We show that for the one-sided Dyck shift one of these measures is the unique tail-invariant probability (section 3). We also characterize the double-tail invariant probabilities for the Dyck shifts

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(section 4). In addition to its two equilibrium measures, the two sided Dyck shift has another double-tail invariant probability – shift invariant, non-equilibrium. These are the only three double-tail invariant, ergodic probabilities on the two sided Dyck shift. A different but perhaps related study of the Dyck shift was carried out by Hamachi and Inoue [9].

2 Definition of The Dyck System

Let us explicitly describe the Dyck language and it's cover (Fischer automaton). Let $m \geq 1$ and $\Sigma = \{\alpha_j : 1 \leq j \leq m\} \cup \{\beta_j : 1 \leq j \leq m\}$, $\Gamma = \{\alpha_j : 1 \leq j \leq m\}^*$, and with Λ the empty word, $\varphi(a, \alpha_j) = a\alpha_j$,$a \in \Gamma$,

$$\varphi(a, \beta_j) = \begin{cases} \beta_j & \text{if } a = \Lambda\text{,or } a = (a_j) \\ \Lambda & \text{if } a \in \{a_j : 1 \leq j \leq m\}^k, k \in \mathbb{N},\ a_k \neq \alpha_j \\ (a_i)_{i=1}^{k-1} & \text{if } a \in \{a_j : 1 \leq j \leq m\}^k,\ k > 1, a_k = \alpha_j \end{cases}$$

Another way to describe the Dyck-Shift is in terms of it's syntactic monoid:

Let M be the monoid generated by Σ, with the following relations:

1. $\alpha_j \cdot \beta_j \equiv \Lambda \equiv 1(modM), j = 1, \ldots, m$
2. $\alpha_i \cdot \beta_j \equiv 0(modM), i \neq j$

The *m-Dyck language* is

$$L = \{l \in \Sigma^* : l \not\equiv 0(modM)\}$$

and the corresponding (two sided) m-Dyck subshift is

$$X = \{x \in \Sigma^{\mathbb{Z}} :\ (x_i)_{i=r}^{l} \in L \text{ for all } -\infty < r \leq l < +\infty\}$$

and we will also refer to the *one sided m-Dyck subshift*:

$$Y = \{y \in \Sigma^{\mathbb{N}} :\ (y_i)_{i=r}^{l} \in L \text{ for all } 0 \leq r \leq l < +\infty\}$$

These are indeed subshifts, since we only pose restrictions on finite blocks. Conversely, we will later note by $L(X) = L(Y) = L$ the language consisting of words which are admissible in X. Also, let:

$$\mathcal{L}_n = \mathcal{L}(Y, n) := L(Y) \cap \Sigma^n$$

Note that when $m = 1$, X is simply the full 2-Shift, and so we will only be interested in the case where $m \geq 2$.

For $w = (w_0, \ldots, w_{n-1}) \in \mathcal{L}(X, n)$ define

$$H(w) = \sum_{i=0}^{n-1} \sum_{j=1}^{m} (\delta_{\alpha_j, w_i} - \delta_{\beta_j, w_i})$$

and $H(\Lambda) = 0$.

For $x \in X$, let

$$H_i(x) = \begin{cases} \sum_{j=0}^{i-1} \sum_{l=1}^{m} (\delta_{\alpha_l, x_j} - \delta_{\beta_l, x_j}) & \text{if } i > 0 \\ \sum_{j=i}^{-1} \sum_{l=1}^{m} (\delta_{\beta_l, x_j} - \delta_{\alpha_l, x_j}) & \text{if } i < 0 \\ 0 & \text{if } i = 0 \end{cases} \tag{1}$$

We shall use the same notation for the one-sided subshift. For $y \in Y$, let

$$H_i(y) = \begin{cases} \sum_{j=0}^{i-1} \sum_{l=1}^{m} (\delta_{\alpha_l, y_j} - \delta_{\beta_l, y_j}) & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$$

where it is clear from the context whether we are refereing to the one sided or two sided subshift. If $w \equiv 1 \pmod{M}$ we say that w is a *balanced word*.
A word w is a *Dyck word* if it is a minimal balanced word. This means $w = \alpha_i \tilde{w} \beta_i$ for some balanced word $\tilde{w}$ and $1 \le i \le m$.

For $w \in \mathcal{L}_n$ define

$$m(w) = \min\{H(u) : \ u \text{ is a prefix of } w\}$$

$$\hat{\alpha}(w) := H(w) - m(w)$$

$$\hat{\beta}(w) := -m(w)$$

where in the definition of $m(w)$ it is understood that the empty word is a prefix of any word, so that $m(w) \le 0$. $\hat{\alpha}$ is the number of unmatched α's in w, and $\hat{\beta}$ is the number of unmatched β's in w. We say that w has an unmatched α at location t if $w_t = \alpha_i$, and $\hat{\alpha}(w_0^{t-1}) < \hat{\alpha}(w_0^t)$. We define "unmatched β" respectively using $\hat{\beta}$. We say that $x \in X$, has an unmatched α (β) at location t if $x_t = \alpha_i$ ($x_t = \beta_i$) which is unmatched in any finite word $x_{[a,b]}$ with $t \in [a, b]$.

2.1 Classification of the Dyck System

Before stating and proving the result regarding invariant measures for the Dyck system, we characterize this subshift in terms of the classes of subshifts introduced in [2],[8], and [5]. The purpose of this subsection is to put in broader context the Dyck shift and the results in the following sections. Detailed discussions of these classes of subshifts can be found in the references above.
By defining the Dyck language as the language recognized by a Fischer automaton, we showed that the Dyck system is a coded system (as in [2]). We claim that the Dyck system is half-synchronized, yet not synchronized (as in [8]):

Proposition 2.1 *Every word w in the Dyck langauge is half synchronizing*

Proof: Suppose $w = w_0, \ldots, w_{n-1}$ Let $(u_k)_{k \in \mathbb{N}}$ be an enumeration of the Dyck words. We define a left infinite sequence $x \in X$ as the word w (ending in coordinate 0), preceded by a concatenation of the words $(u_k)_{k \in \mathbb{N}}$, and followed by

an infinite sequence of α_j's. x is a left-transitive point. $\omega_+(x(-\infty,0]) = \omega + w$, since every unmatched α_j in $x(-\infty,0)]$ must be in w. $\square$

Proposition 2.2 *The m-Dyck system is not a synchronized system, for $m > 1$.*

Proof: Let $w \in L(X)$. There exist l, r such that $lwr \equiv 1 \pmod{M}$. Thus, for $i \neq j$, $\alpha_i lwr\beta_j \notin L(X)$, but $\alpha_i lw \in L(X)$ and $wr\beta_j \in L(X)$. This show that w is not a synchronizing word. $\square$

In [5] Buzzi, defined and studied a class of subshifts called *subshifts of quasi-finite type*. We state without proof the following:

Proposition 2.3 *For $m > 1$, the m-Dyck system is not weak quasi-finite type.*

2.2 Maximal Measures for the Dyck Shift

In [11] Krieger introduced the following decomposition of X into shift invariant subsets:

$$A_+ = \{y \in X : \lim_{i\to\infty} H_i(y) = -\lim_{i\to-\infty} H_i(y) = \infty\}$$

$$A_- = \{y \in X : -\lim_{i\to\infty} H_i(y) = \lim_{i\to-\infty} H_i(y) = \infty\}$$

$$A_0 = \bigcap_{i=-\infty}^{\infty} (\bigcup_{l=1}^{\infty} \{y \in X : H_i(y) = H_{i+l}(y)\} \cap \bigcup_{l=1}^{\infty} \{y \in X : H_i(y) = H_{i-l}(y)\})$$

Since the complement of these sets, $X \backslash (A_+ \cup A_- \cup A_0)$ is a countable union of wandering sets, every ergodic shift-invariant probability measure assigns probability one to exactly one of these sets.
Let further

$$B_+ = \bigcap_{i=-\infty}^{\infty} (\bigcup_{l=1}^{m} (\{x \in X : x_i = \alpha_l\} \cup \bigcup_{k=1}^{\infty} \{x \in X : x_i = \beta_l, H_{i-k}(x) = H_i(x)\}))$$

$$B_- = \bigcap_{i=-\infty}^{\infty} (\bigcup_{l=1}^{m} (\{x \in X : x_i = \beta_l\} \cup \bigcup_{k=1}^{\infty} \{x \in X : x_i = \alpha_l, H_{i+k}(x) = H_i(x)\}))$$

and observe that $A_+ \cup A_0 \subset B_+$, $A_- \cup A_0 \subset B_-$. Let $\Omega = \{\alpha_1 \ldots \alpha_m, \beta\}^{\mathbb{Z}}$. Define $\widehat{H}_0(x) = 0$, $\widehat{H}_i(x) = \sum_{j=0}^{i-1} (\sum_{k=1}^{m} \delta_{x_j,\alpha_k} - \delta_{x_j,\beta})$, $x \in \Omega$. Denote

$$\widehat{B}_+ = \bigcap_{i=-\infty}^{\infty} (\bigcup_{l=1}^{m} \{\omega \in \Omega : \omega_i = \alpha_l\} \cup \bigcup_{k=1}^{\infty} \{\omega \in \Omega : \omega_i = \beta, \widehat{H}_{i-k}(\omega) = \widehat{H}_i(\omega)\})$$

$$\widehat{A}_+ = \{\omega \in \Omega : \lim_{i\to\infty} \widehat{H}_i(\omega) = -\lim_{i\to-\infty} \widehat{H}_i(\omega) = \infty\}$$

$$\widehat{A}_0 = \bigcap_{i=-\infty}^{\infty} (\bigcup_{l=1}^{\infty} \{\omega \in \Omega : \widehat{H}_i(\omega) = \widehat{H}_{i+l}(\omega)\} \cap \bigcup_{l=1}^{\infty} \{\omega \in \Omega : \widehat{H}_i(\omega) = \widehat{H}_{i-l}(\omega)\})$$

$$(g_+(y))_i = \begin{cases} \alpha_j & y_i = \alpha_j \\ \beta & y_i \in \{\beta_1, \ldots, \beta_m\} \end{cases}$$

g_+ is a one-to-one Borel mapping from B_+ onto $\widehat{B}_+$, commuting with the shift. This shows that every shift invariant probability measure μ on X such that $\mu(B_+) = 1$ can be transported to a shift invariant probability on Ω with equal entropy. By the intrinsic ergodicity of the full-shift, there is a unique measure μ_1 of maximal entropy on X such that $\mu_1(B_+) = 1$. This measure is supported by $A+ \subset B_+$. By similar arguments, there is a unique measure μ_2 of maximal entropy on X such that $\mu_2(B_-) = 1$, and in fact $\mu_2(A_-) = 1$.

Remark 2.1

$$\sup_{\mu \in \mathcal{P}(A_0, T)} \{h(A_0, T, \mu)\} = \log(2) + \frac{1}{2} \log m$$

Proof: Since $A_0 \subset B_+$, any shift invariant probability μ_0 on X supported by A_0 can also be transported to a probability $\widehat{\mu}_0$ on Ω via g_+. By the ergodicity, $\widehat{\mu}_0([\beta]) = \frac{1}{2}(1 - \lim_{n\to\infty} \frac{H_n(x)}{n}) = \frac{1}{2}$. Thus, $h(X, T, \mu_0) = h(\Omega, T, \widehat{\mu}_0) \leq \log(2) + \frac{1}{2}\log(m)$, and equality can be obtained by taking $\widehat{\mu}_0 = \prod_{i=-\infty}^{+\infty} (\frac{1}{2m}, \ldots, \frac{1}{2m}, \frac{1}{2})$. $\square$

3 Tail Invariant Measures For One Sided Dyck Shift

In this section we consider the one sided Dyck shift. We prove the following result:

Theorem 3.1 *The tail relation of the one sided Dyck shift is uniquely ergodic. Furthermore, there exists a unique topologically σ-finite tail-invariant measure on the one sided Dyck shift (up to multiplication by a positive real number).* [2]

This theorem is a direct conclusion of lemmas 3.3, 3.4 and lemma 3.5 below.

Lemma 3.1 *The tail relation of the one sided m-Dyck is topologically transitive.*

[2]A measure μ on topological space X *is topologically σ-finite* if there is a countable cover of $X \setminus N$ by open sets with finite μ-measure, where N is a μ-null set.

Proof: Let $y = (y_n) \in \{\alpha_1, \ldots, \alpha_m\}^{\mathbb{N}} \subset Y$. To prove the lemma, we will show that $\mathcal{T}(y)$ is dense in Y. Let $\omega = (\omega_1, \ldots, \omega_r) \in L(Y)$, then $wy_{r+1}^{\infty} \in Y$. Thus, $[w] \cap \mathcal{T}(y) \neq \emptyset$. This proves $\overline{\mathcal{T}(Y)} = Y$. $\square$

If a tail-invariant measure μ on Y is topologically σ-finite, $\exists w \in L(Y)$ s.t. $0 < \mu([w]) < \infty$. A corollary of our main result is that any such μ is a finite measure.

Define the following tail-invariant decomposition of the one-sided Dyck shift:

$$G_+ = \{y \in Y : \lim_{i\to\infty} H_i(y) = +\infty\}$$

$$G_- = \{y \in Y : \liminf_{i\to\infty} H_i(y) = -\infty\}$$

$$G_0 = \{y \in Y : \liminf_{i\to\infty} H_i(y) \in (-\infty, +\infty)\}$$

Obviously, $Y = G_+ \uplus G_- \uplus G_0$

Let

$$W_n = W_n^m := \{l \in \mathcal{L}(Y, n) :\ l \equiv 1 (mod M)\}$$

where M is the syntactic monoid of the m-Dyck shift. W_n is the set of balanced words of length n. Denote:

$$w_n = w_n^m := |W_n^m|$$

Let

$$\widetilde{W}_n^m = \widetilde{W}_n = \{l \in \mathcal{L}(Y, n) :\ l = \alpha_i \tilde{l} \beta_i\ , 1 \leq i \leq m\ , \tilde{l} \in W_{n-2}\}$$

$\widetilde{W}_n$ is the set of Dyck words of length n. Denote $\tilde{w}_n = \tilde{w}_n^m := |\widetilde{W}_n^m|$. Obviously, $\tilde{w}_n \leq w_n$

Lemma 3.2

$$w_{2k}^m = \frac{\binom{2k}{k}}{k+1} m^k$$

Proof: First, we note that $w_{2k}^m = m^k w_{2k}^1$. This follows from the fact that given $a \in W_{2k}^1$ one can independently choose the "type" of each pair of brackets in order to create distinct elements in W_{2k}^m, and every element of W_{2k}^m can be created this way. This describes a m^k to one mapping $W_{2k}^m \mapsto W_{2k}^1$.
All that remains is to prove

$$w_{2k}^1 = \frac{\binom{2k}{k}}{k+1}$$

This is sometimes called *the ballot problem.* An elementary proof of this can be found in pages 69-73 of [7]. $\square$

Lemma 3.3 *There are no topologically σ-finite tail invariant measures, giving G_0 positive measure.*

Proof: Suppose μ is a tail invariant measure s.t. $0 < \mu(G_0 \cap [v]) < \infty$ and $|v| = l$. Without loss of generality, we can assume $\mu(G_0 \cap [v]) = 1$. Let R_n be the subset of X consisting of points which are balanced from time n onwards:

$$R_n = \{y \in Y : \forall l \geq n \ H_l(y) \geq H_n(y), \ \liminf_{i \to \infty} H_i(y) = H_n(y)\}$$

We write the following decomposition of $[v] \cap R_n$, according to the first Dyck word following v:

$$\forall n \geq l, \ \ [v] \cap R_n = \biguplus_k \biguplus_{w \in \widetilde{W}_k} (T^{-n}[w] \cap R_n) \cap [v] = \biguplus_k \biguplus_{w \in \widetilde{W}_k} (T^{-n}[w] \cap R_{n+k}) \cap [v]$$

We further decompose each of these sets:

$$[v] \cap T^{-n}[w] \cap R_{n+k} = \biguplus_{a \in \mathcal{L}_n} (T^{-n}[w] \cap R_{n+k} \cap [a]) \cap [v] = \biguplus_{a \in \mathcal{L}_n, a_1^l = v} [aw] \cap (R_{n+k})$$

We note that

$$R_{n+k} \cap [v] = \biguplus_{b \in \mathcal{L}(Y, n+k), a_1^l = v} R_{n+k} \cap [b]$$

and for every $a, b \in \mathcal{L}_{n+k}$, $\mu(R_{n+k} \cap [a]) = \mu(R_{n+k} \cap [b])$ because μ is $\mathcal{T}(Y)$-invariant.

Let $r_n = \mu(R_n \cap [v])$, $a_n = |\{\alpha \in \mathcal{L}_n : \alpha_1^l = v\}|$,$r_\infty = \sup_n r_n$, then:

$$r_n = \sum_k \frac{a_n \tilde{w}_k}{a_{n+k}} r_{n+k}$$

so:

$$r_\infty \leq \sup_{n \geq 0} (\sum_k \frac{a_n w_k}{a_{n+k}}) r_\infty$$

Since $r_n \leq \mu([v] \cap G_0)$ and $G_0 = \bigcup_n R_n$, we have $0 < r_\infty < \infty$ .We obtain:

$$1 \leq \sup_{n \geq 0} \sum_k \frac{a_n w_k}{a_{n+k}}$$

Since for any $u \in \mathcal{L}_n$ and any $1 \leq j \leq m$, $u\alpha_j \in \mathcal{L}_{n+1}$ and $\exists 1 \leq j \leq m$ $u\beta_j \in \mathcal{L}_{n+1}$, we get the inequality $\frac{a_{n+1}}{a_n} \geq m + 1$. This proves $\frac{a_n}{a_{n+2k}} \leq \frac{1}{(m+1)^{2k}}$. Also,

$$w_{2k} = \frac{\begin{pmatrix} 2k \\ k \end{pmatrix}}{k+1} m^k$$

from this follows:

$$\sum_{k=1}^{\infty} \frac{a_n w_k}{a_{n+k}} \leq \sum_{k=1}^{\infty} \frac{\begin{pmatrix} 2k \\ k \end{pmatrix}}{k+1} \left(\frac{m}{(m+1)^2} \right)^k$$

but:

$$\sum_{k=1}^{\infty} \frac{\begin{pmatrix} 2k \\ k \end{pmatrix}}{k+1} x^k = x^{-1} \int_0^x \sum_{k=1}^{\infty} \begin{pmatrix} 2k \\ k \end{pmatrix} t^k dt = \frac{1-\sqrt{1-4x}}{2x} - 1$$

so for $m > 1$:

$$\sum_{k=1}^{\infty} \frac{a_n w_k}{a_{n+k}} \leq \frac{(m+1)^2}{2m} \left(1 - \frac{m-1}{m+1}\right) - 1 = \frac{1}{m}$$

which implies:

$$\sup_{n \geq 0} \sum_k \frac{a_n w_k}{a_{n+k}} \leq \frac{1}{m} < 1$$

This gives us a contradiction to our assumption of the existence of such a measure μ. $\square$

Lemma 3.4 *There are no topologically σ-finite tail invariant measures, giving G_+ positive measure.*

Proof: Assume there exist a tail invariant measure μ such that $0 < \mu(G_+ \cap [v]) < \infty$. Since G_+ is a tail invariant subset, we can assume $\mu(G_+^c) = 0$ by taking $\mu'(A) = \mu(A \cap G_+)$. Let:

$$\tilde{F}_n = \bigcap_{k>n} \{H_k(y) > H_n(y)\}$$

and:

$$F_n = \tilde{F}_n \setminus \bigcup_{j=1}^{n-1} \tilde{F}_j$$

F_n is the sets of points which have the first α which is unmatched at coordinate n. By definition,

$$G_+ \subset \bigcup_{n>0} F_n$$

so for some n we must have $0 < \mu(F_n \cap [v]) < \infty$.

$$F_n = \bigcup_w \bigcup_{i=1}^{m} (F_n \cap [w\alpha_i])$$

where the union is over all $w \in \mathcal{L}_{n-1}$ with $\hat{\alpha}(w) = 0$. For any $K \in \mathbb{N}$ we have that:

$$(F_n \cap [w\alpha_i]) = \biguplus_b (F_n \cap [w\alpha_i b])$$

where this time the union is over all $b \in \mathcal{L}_K$ such that $\hat{\beta}(b) = 0$. The reason there should be no unmatched β's in b is so they will not match the α_i at coordinate n. We denote the set of such b's by U_K. Suppose such b has $\hat{\alpha}(b) = j$ with $j > 0$. Denote by $\xi(b,t)$, $0 < t < j-1$ (which also depends on w), the word obtained from b by replacing the leftmost unmatched α_s by β_i (so as to match the unmatched α_i) and replacing the next t leftmost unmatched α_s with β_s. It follows from the construction that for any $y \in Y$, if $w\alpha_i b y \in Y$ then $w\alpha_i \xi(b,t) y \in Y$.

This shows there is a tail holonomy $\pi : [w\alpha_i b] \to \pi[w\alpha_i b] \subset [w\alpha_i \xi(b,t)]$, so $\mu([w\alpha_i b]) \leq \mu([w\alpha_i \xi(b,t)])$. For $b_1, b_2 \in U_K$, if $\xi(b_1,t) = \xi(b_2,t)$, this implies that b_1 and b_2 can differ only where the first unmatched α is located - so the maps $\xi(.,t)$ are m to 1. Let

$$C(K,n,j) := \bigcap_{N>K} \{H_N(y) > j + H_{n+1}(y)\}$$

By definition, $G_+ \subset \bigcup_K C(K,n,j)$. From our assumption that μ is supported on G_+, it follows that $\mu((\bigcup_K C(K,n,j))^c) = 0$. Since $C(K,n,j)$ is an increasing sequence of sets, there exist K_0 such that

$$\mu(C(K_0,n,j) \cap [w\alpha_i] \cap F_n) > (1 - \frac{1}{j})\mu([w\alpha_i] \cap F_n)$$

define:

$$U(K_0,j) = \{b \in \mathcal{L}_{K_0} : \ \hat{\alpha}(b) > j, \ \hat{\beta}(b) = 0\}$$

Note that if $t_1 \neq t_2$, then $\xi(b_1,t_1) \neq \xi(b_2,t_2)$, because they have different number of unmatched β's. We have:

$$C(K_0,n,j) \cap [w\alpha_i] \cap F_n = \biguplus_{b \in U(K_0,j)} ([w\alpha_i b]) \cap C(K_0,n,j) \cap F_n$$

For the above K_0, the following inequalities hold:

$$\mu(F_n \cap [w\alpha_i]) \leq \frac{j}{j-1}\mu(F_n \cap [w\alpha_i] \cap C(K_0,n,j)) =$$

$$\sum_{b \in U(K_0,j)} \mu([w\alpha_i b]) \cap C(K_0,n,j) \cap F_n) \leq$$

$$\sum_{b \in U(K_0,j)} \mu([w\alpha_i b]) \cap C(K_0,n,j)) \leq$$

Because $\xi(.,t)$ are m to 1:

$$\leq m \sum_{b \in U(K_0,j)} \mu([w\alpha_i \xi(b,t)])$$

We average this in equality over t:

$$\mu(F_n \cap [w\alpha_i]) \leq \frac{m}{j-1} \sum_{t=1}^{j-1} \sum_{b \in U(K_0,j)} \mu([w\alpha_i \xi(b,t)])$$

because:

$$[w\alpha_i] \supseteq \biguplus_{t=1}^{j-1} \biguplus_{b\in U(K_0,j)} ([w\alpha_i\xi(b,t)])$$

we obtain:

$$\mu(F_n \cap [w\alpha_i]) \leq \frac{m}{j-1}\mu([w\alpha_i])$$

We assume $\mu([w\alpha_i]) \leq \mu([v]) = \mu(G_+ \cap [v]) < \infty$, so taking $j \to \infty$ we obtain that $\mu(F_n \cap [w\alpha_i]) = 0$, and since $G_+ \cap [v]$ is a countable union of such sets we conclude that $\mu(G_+ \cap [v]) = 0$. □

We conclude that every tail invariant measure of the Dyck shift is supported by

$$G_- = \{y \in X : \liminf_{i\to\infty} H_i(y) = -\infty\}$$

To prove unique ergodicity, we need the following:

Lemma 3.5 *There exists a unique tail-invariant probability measure μ on Y such that $\mu(G_-) = 1$. Furthermore, for any topologically σ-finite tail-invariant measure μ' on G_-, $\mu' = c\mu$ for some positive number c.*

Proof: Let $\Theta = \{\beta_1, ..., \beta_m, \alpha\}^{\mathbb{N}}$. Define $\widetilde{H}_0(x) = 0$, $\widetilde{H}_i(x) = \sum_{j=1}^{i}(-\sum_{k=1}^{m}\delta_{x_j,\beta_k} + \delta_{x_j,\alpha})$, $x \in \Theta$. Denote

$$\Theta_- = \{x \in \Theta : \liminf_{i\to\infty} \widetilde{H}_i(x) = -\infty\}$$

We will use a one-to-one Borel mapping of G_- on to Θ_-, introduced in [11]. The map is defined is follows:

$$g_- : G_- \to \Theta_-$$

$$g_-(y)_i = \begin{cases} \alpha & y_i \in \{\alpha_1, \ldots, \alpha_m\} \\ \beta_j & y_i = \beta_j \end{cases}$$

g_- is a bijection, and for any $y_1, y_2 \in G_-$ $(y_1, y_2) \in \mathcal{T}(Y) \Leftrightarrow (g_-(y_1), g_-(y_2)) \in \mathcal{T}(\Theta)$. Let p be the symmetric Bernoulli measure on Ω satisfying $p([\omega_1, \ldots, \omega_n]) = (\frac{1}{m+1})^n$. by the law of large numbers $p(\Theta_-) = 1$, and therefor $p \circ g_-(G_-) = 1$. So $p \circ g_-$ is a tail invariant probability measure on Y supported by G_-. Suppose μ is a tail invariant probability measure on Y s.t. $\mu(G_-) = 1$. μ can be transported by g_- to a tail invariant probability measure q on Θ (supported by Θ_-). Since Θ is a full-shift, the uniqueness of $\mathcal{T}(\Theta)$-invariant topologically σ-finite measure follows immediately from the fact that all cylinders of the same length have equal measure. This proves the uniqueness of a tail-invariant topologically σ-finite measure on G_-. □

4 Two Sided Dyck Shift

4.1 Maximal Entropy Implies Double-Tail Invariance

In [11] it was demonstrated that the Dyck shift has two ergodic shift invariant probabilities with entropy equal to the topological entropy. Such probabilities are called *equilibrium states*. In this section we show that both of these probabilities are also double-tail invariant.

We introduce the following sets, which are mutually disjoint and are double-tail invariant. For $s, t \in \{\{+\infty\}, \{-\infty\}, \mathbb{R}\}$ we define:

$$B_t^s = \{x \in X : \liminf_{i \to +\infty} H_i(x) \in s, \liminf_{i \to -\infty} H_i(x) \in t\}$$

let

$$\Omega_-^+ = \{x \in \{\alpha_1 \dots \alpha_m, \beta\}^{\mathbb{Z}} : \liminf_{i \to +\infty} \widehat{H}_i(x) = +\infty, \liminf_{i \to -\infty} \widehat{H}_i(x) = -\infty\}$$

and

$$\Theta_+^- = \{x \in \{\beta_1 \dots \beta_m, \alpha\}^{\mathbb{Z}} : \liminf_{i \to +\infty} \widetilde{H}_i(x) = -\infty, \liminf_{i \to -\infty} \widetilde{H}_i(x) = +\infty\}$$

Where $\widehat{H}$ and $\widetilde{H}$ are defined on $\{\alpha_1 \dots \alpha_m, \beta\}^{\mathbb{Z}}$ and $\{\beta_1 \dots \beta_m, \alpha\}^{\mathbb{Z}}$ respectively, as in formula (1).

Define:

$$g_+ : B_{-\infty}^{+\infty} \mapsto \Omega_{-\infty}^{+\infty}$$

$$(g_+(y))_i = \begin{cases} \alpha_j & y_i = \alpha_j \\ \beta & y_i \in \{\beta_1, \dots, \beta_m\} \end{cases}$$

$$g_- : B_{+\infty}^{-\infty} \mapsto \Theta_{+\infty}^{-\infty}$$

$$(g_-(y))_i = \begin{cases} \beta_j & y_i = \beta_j \\ \alpha & y_i \in \{\alpha_1, \dots, \alpha_m\} \end{cases}$$

g_+ is a Borel bijection from $B_{-\infty}^{+\infty}$ to $\Omega_{-\infty}^{+\infty}$ and g_- is a Borel bijection of the appropriate sets. The definitions of g_+ and g_- can also be extended to functions $g_+ : B_{-\infty}^{\mathbb{R}} \mapsto \Omega_{-\infty}^{\mathbb{R}}$ and $g_- : B_{\mathbb{R}}^{-\infty} \mapsto \Theta_{\mathbb{R}}^{-\infty}$, which are also Borel bijections.

Lemma 4.1 *$g_+ : B_{-\infty}^{+\infty} \mapsto \Omega_{-\infty}^{+\infty}$, $g_- : B_{+\infty}^{-\infty} \mapsto \Theta_{+\infty}^{-\infty}$, $g_+ : B_{-\infty}^{\mathbb{R}} \mapsto \Omega_{-\infty}^{\mathbb{R}}$, $g_- : B_{\mathbb{R}}^{-\infty} \mapsto \Theta_{\mathbb{R}}^{-\infty}$ are isomorphisms of the two sided tail relations:*

$$(g_+ \times g_+)(\mathcal{T}_2(B_{-\infty}^{+\infty})) = \mathcal{T}_2(\Omega_{-\infty}^{+\infty})$$

$$(g_- \times g_-)(\mathcal{T}_2(B_{+\infty}^{-\infty})) = \mathcal{T}_2(\Theta_{+\infty}^{-\infty})$$

$$(g_+ \times g_+)(\mathcal{T}_2(B_{-\infty}^{\mathbb{R}})) = \mathcal{T}_2(\Omega_{-\infty}^{\mathbb{R}})$$

$$(g_- \times g_-)(\mathcal{T}_2(B_{\mathbb{R}}^{-\infty})) = \mathcal{T}_2(\Theta_{\mathbb{R}}^{-\infty})$$

Proof: We prove the result for $g_+ : B_{-\infty}^{+\infty} \mapsto \Omega_{-\infty}^{+\infty}$, the other results are proved in the same manner.
$(g_+ \times g_+)(\mathcal{T}_2(B_{-\infty}^{+\infty})) \subset \mathcal{T}_2(\Omega_{-\infty}^{+\infty})$ is trivial, so we show the other inclusion. Suppose $(g_+(x), g_+(y)) \in \mathcal{T}_2(\Omega_{-\infty}^{+\infty})$. Let $n_0 \geq 0$ be such that $g_+(x)_{[-n_0,n_0]^c} = g_+(y)_{[-n_0,n_0]^c}$.
Let

$$r(i,x) = max\{j < i : H_j(x) = H_i(x)\}$$

Clearly, $r(i_1, x) = r(i_2, x)$ is impossible for $i_1 \neq i_2$. Since

$$\liminf_{n\to+\infty} H_n(x), \liminf_{n\to+\infty} H_n(y) > -\infty$$

there exists c such that for some large N, $H_i(x) > c$ for every $i > N$. Since $\liminf_{n\to-\infty} H_n(x) = \liminf_{n\to-\infty} H_n(y) = -\infty$, it follows that there exist some $i_0 < N$ such that $H_{i_0}(x) = c$, so for every $i > N$, $r(i,x) > i_0$. The same argument applies for y. Since $(r(i,x))_{i>N}$ and $((r(i,y))_{i>N}$ are both injective sequences of integers, bounded from below, it follows that

$$\lim_{n\to+\infty} r(n,x) = \lim_{n\to+\infty} r(n,y) = +\infty$$

Note that for $n_1, n_2 > n_0$,

$$\widehat{H}_{n_1}(g_+(x)) - \widehat{H}_{n_2}((g_+(x)) = \widehat{H}_{n_1}(g_+(y)) - \widehat{H}_{n_2}((g_+(y))$$

so for all large n enough so that $r(n,x) > n_0$,$r(n,y) > n_0$, there are exactly two cases:

1. $g_+(x)_n = g_+(y)_n = \beta$,in which case $r(n,x) = r(n,y)$ and $x_{r(n,x)} = y_{r(n,y)}$, so $x_n = y_n$
2. $g_+(x) = g_+(y) = \alpha_i$ for $1 < i < m$, and then $x_n = y_n = \alpha_i$

Obviously, for $n < -n_0$, $x_n = y_n$. This proves $(x,y) \in \mathcal{T}_2(B_{-\infty}^{+\infty})$. □

Lemma 4.2 *There exists a unique $\mathcal{T}_2$-invariant probability of X supported by $B_{-\infty}^{+\infty}$, and a unique $\mathcal{T}_2$-invariant probability of X supported by $B_{+\infty}^{-\infty}$. There are no $\mathcal{T}_2$-invariant probabilities on $B_{-\infty}^{\mathbb{R}}$ and $B_{\mathbb{R}}^{-\infty}$.*

Proof: The symmetric product measure p on Ω assigns probability one to Ω_-^+ Transporting the product measure on Ω by means of g_+^{-1} to $B_{-\infty}^{+\infty}$ yields a tail invariant probability measure on X, by the previous lemma.
On the other hand, any tail invariant probability on X supported by $B_{-\infty}^{+\infty} \cup B_{-\infty}^{\mathbb{R}}$ can be transported to a tail invariant probability on Ω by g_+. This is an injective correspondence, so by the uniqueness of double-tail invariant probability on Ω, we conclude the uniqueness of double-tail invariant probability on $B_{-\infty}^{+\infty} \cup B_{-\infty}^{\mathbb{R}}$. In particular, this also proves that no double-tail invariant probability on $B_{-\infty}^{\mathbb{R}}$ exist. We obtain the results for $B_{+\infty}^{-\infty}$ and $B_{\mathbb{R}}^{-\infty}$ symmetrically. □

4.2 A Third Double-tail Invariant Probability

For $z \in \{0,1\}^{\mathbb{Z}}$, we define:

$$\tilde{H}_i(z) = \begin{cases} \sum_{j=0}^{i-1}(\delta_{1,z_j} - \delta_{0,z_j}) & \text{if } i > 0 \\ \sum_{j=i}^{-1}(\delta_{0,z_j} - \delta_{1,z_j}) & \text{if } i < 0 \\ 0 & \text{if } i = 0 \end{cases}$$

Let

$$S_{-\infty}^{-\infty} = \{z \in \{0,1\}^{\mathbb{Z}} : \inf_{n \geq 0} \tilde{H}_n(z) = -\infty \ , \inf_{n<0} \tilde{H}_n(z) = -\infty \ \}$$

Let us define a Borel function $F : S_{-\infty}^{-\infty} \times \{1,\ldots,m\}^{\mathbb{Z}} \mapsto \Sigma^{\mathbb{Z}}$:

Let

$$F(z,a)_n = \begin{cases} \alpha_j & \text{if } z_n = 1,\ a_{\gamma_n(z)} = j \\ \beta_j & \text{if } z_n = 0,\ k = \varepsilon_n(z), \text{ and } a_{\gamma_k(z)} = j \end{cases}$$

where,

$$\gamma_k(z) = \begin{cases} \sum_{i=0}^{k} z_i & k \geq 0 \\ -\sum_{i=k}^{-1} z_i & k < 0 \end{cases}$$

$$\varepsilon_n(z) = \max\{l < n : \ \tilde{H}_l(z) \leq \tilde{H}_{n+1}(z)\}$$

Since $\liminf_{n \to -\infty} \tilde{H}_n(z) = -\infty$ for $z \in S_{-\infty}^{-\infty}$, F is well defined.

Lemma 4.3 *For every $z \in S_{-\infty}^{-\infty}$, $a \in \{1,\ldots,m\}^{\mathbb{Z}}$, $F(z,a) \in X$.*

Proof: Suppose $x = F(z,a) \notin X$, then there exist $n, n' \in \mathbb{Z}$, $n < n'$, such that $x_n = \alpha_i$, $x_{n'} = \beta_j$ with $i \neq j$ and $n = \max\{l < n' : \ H_l(x) = H_{n'+1}(x)\}$. But in that case, $n = \varepsilon_{n'}(z)$, so $i = j = a_{\gamma_n(z)}$. □

Let μ_1 be the symmetric product measure on $\{0,1\}^{\mathbb{Z}}$, and μ_2 the symmetric product measure on $\{1,\ldots,m\}^{\mathbb{Z}}$.

Lemma 4.4 $\mu_1(S_{-\infty}^{-\infty}) = 1$

Proof: This follows from recurrence and ergodicity of the simple random walk on $\mathbb{Z}$. □

We define: $\tilde{\mu} = (\mu_1 \times \mu_2) \circ F^{-1}$. Since $F^{-1}(B_{-\infty}^{-\infty}) = S_{-\infty}^{-\infty} \times \{1,\ldots,m\}^{\mathbb{Z}}$ it follows that $\tilde{\mu}(B_{-\infty}^{-\infty}) = 1$.

Let us also define a Borel mapping $z : B_{-\infty}^{-\infty} \mapsto S_{-\infty}^{-\infty}$:

$$z(x)_n = \begin{cases} 1 & x_n \in \{\alpha_1,\ldots,\alpha_m\} \\ 0 & x_n \in \{\beta_1,\ldots,\beta_m\} \end{cases}$$

The following lemma gives an explicit formula for the $\tilde{\mu}$ probability of a cylinder:

Lemma 4.5 *Let $w \in L(X)$. If the number of matched α's in w is n_1 and the number of unmatched α's and β's is n_2 ($2n_1 + n_2 = |w|$) then $\tilde{\mu}([w]_k) = m^{-(n_1+n_2)}(\frac{1}{2})^{|w|}$.*

Proof: Denote by $f_1, \ldots, f_{n_1}$ the locations of matched α's in w. Denote by $g_1, \ldots, g_{n_2'}$ the locations of unmatched α's in w. Denote by $h_1, \ldots, h_{n_2''}$ the locations of unmatched β's in w. We have $n_2' + n_2'' = n_2$. For $\vec{r} \in \mathbb{Z}^{n_1}$, $\vec{s} \in \mathbb{Z}^{n_2'}$, $\vec{t} \in \mathbb{Z}^{n_2''}$, define:

$$A_{\vec{r}} = \{z : \ \gamma_{k+f_l}(z) = r_l \ 1 \leq l \leq n_1\}$$
$$B_{\vec{s}} = \{z : \ \gamma_{k+g_l}(z) = s_l \ 1 \leq l \leq n_2'\}$$
$$C_{\vec{t}} = \{z : \ \gamma_{\varepsilon_l}(z) = t_l \ \varepsilon_l = \varepsilon_{k+h_l}(z) 1 \leq l \leq n_2''\}$$

Informally, $A_{\vec{r}}, B_{\vec{s}}, C_{\vec{t}}$ determine the locations in the sequence $a \in \{1, \ldots, m\}^{\mathbb{Z}}$ involved in selecting the types of α's and β's within the coordinates $k, \ldots, k+|w|$. Now we define:

$$Z = \{z \in S_{-\infty}^{-\infty} : \ z_{i+k} = z(w)_i \text{ for } 0 \leq i \leq |w|\}$$

$$A'_{\vec{r}} = \{a \in \{1, \ldots, m\}^{\mathbb{Z}} : \ a_{r_l} = j \text{ if } w_{f_l} = \alpha_j\}$$
$$B'_{\vec{s}} = \{a \in \{1, \ldots, m\}^{\mathbb{Z}} : \ a_{s_l} = j \text{ if } w_{g_l} = \alpha_j\}$$
$$C'_{\vec{t}} = \{a \in \{1, \ldots, m\}^{\mathbb{Z}} : \ a_{t_l} = j \text{ if } w_{h_l} = \beta_j\}$$

With the above definitions we can write:

$$F^{-1}([w]_k) = Z \times \{1, \ldots, m\}^{\mathbb{Z}} \cap \bigcup_{\vec{s},\vec{t},\vec{r}} ((A_{\vec{r}} \times A'_{\vec{r}}) \cap (B_{\vec{s}} \times B'_{\vec{s}}) \cap (C_{\vec{t}} \times C'_{\vec{t}})) \quad (2)$$

Where the union of $\vec{r}, \vec{s}, \vec{t}$ ranges over all vectors such that the set of numbers appearing in their coordinates are pairwise disjoint. This is a union of disjoint sets. Thus:

$$\tilde{\mu}([w]_k) = \sum_{\vec{s},\vec{t},\vec{r}} (\mu_1 \times \mu_2)((Z \cap A_{\vec{r}} \cap B_{\vec{s}} \cap C_{\vec{t}}) \times (A'_{\vec{r}} \cap B'_{\vec{s}} \cap C'_{\vec{t}}))$$

$$\tilde{\mu}([w]_k) = \sum_{\vec{s},\vec{t},\vec{r}} \mu_1(Z \cap A_{\vec{r}} \cap B_{\vec{s}} \cap C_{\vec{t}}) \mu_2(A'_{\vec{r}} \cap B'_{\vec{s}} \cap C'_{\vec{t}}) \quad (3)$$

Now notice that for every $\vec{r}, \vec{s}, \vec{t}$ in the sum,

$$\mu_2(A'_{\vec{r}} \cap B'_{\vec{s}} \cap C'_{\vec{t}}) = m^{-(n_1+n_2'+n_2'')} = m^{-(n_1+n_2)}$$

Also note that $Z = \biguplus_{\vec{s},\vec{t},\vec{r}} (Z \cap A_{\vec{r}} \cap B_{\vec{s}} \cap C_{\vec{t}})$, so $\mu_1(Z) = \sum_{\vec{s},\vec{t},\vec{r}} \mu_1(Z \cap A_{\vec{r}} \cap B_{\vec{s}} \cap C_{\vec{t}})$. Thus, equation 3 can be simplified as follows:

$$\tilde{\mu}([w]_k) = \sum_{\vec{s},\vec{t},\vec{r}} \mu_1(Z \cap A_{\vec{r}} \cap B_{\vec{s}} \cap C_{\vec{t}}) m^{-(n_1+n_2)} = \mu_1(Z) m^{-(n_1+n_2)} = (\frac{1}{2})^{|w|} m^{-(n_1+n_2)}$$

$\square$

Theorem 4.1 *$\tilde{\mu}$ is a $\mathcal{T}_2$-invariant probability.*

Our method of proving this is as follows: We define a countable set of $\mathcal{T}_2$-holonomies

$$\mathcal{H} = \{g_{w,w',n} :\ n \in \mathbb{Z},\ w, w' \in L(X)\ |w| = |w'|,\ w \equiv w' (\text{mod } M),\ \}$$

By proposition 4.3 below, we see that $\tilde{\mu}$ is invariant under $\mathcal{H}$. Then we prove that $\mathcal{H}$ generates $\mathcal{T}_2$, up to a $\tilde{\mu}$-null set (proposition 4.4 bellow). This will complete the proof.

Lemma 4.6 *Suppose $w, w' \in \mathcal{L}(X, n)$ with $w \equiv w'$ (mod M). If $x, y \in \Sigma^{\mathbb{Z}}$ such that $x_{[k-n,k]} = w$, $y_{[k-n,k]} = w'$ and $x_{[k-n,k]^c} = y_{[k-n,k]^c}$, then*

$$x \in X \Leftrightarrow y \in X$$

Proof: Suppose $x \in X$. We have to show that for every $j > n$ $y_{[k-j,j]} \not\equiv 0$ (mod M). Writing $x_{[k-j,j]} = swt$, we have $y_{[k-j,j]} = sw't$ and since $w \equiv w'$ (mod M), $sw't \equiv swt \not\equiv 0$ (mod M). This shows $y \in X$. By replacing the roles of y and x we get: $y \in X \Rightarrow x \in X$. $\square$

Let $w, w' \in \mathcal{L}(X, n)$ with $w \equiv w'$ (mod M) and $k \in \mathbb{Z}$. By lemma 4.6 we can define $g_{w,w',k} : [w]_k \mapsto [w']_k$ to be the Borel function that changes the n coordinates starting at k from w to w'.

$$g_{w,w',k}(\ldots, x_{k-1}, w_0, \ldots, w_{n-1}, x_{k+n}, \ldots) = (\ldots, x_{k-1}, w'_0, \ldots, w'_{n-1}, x_{k+n} \ldots)$$

Proposition 4.2 *If $w \equiv w'$(mod M) , $|w| = |w'|$, and $k \in \mathbb{Z}$, then $\tilde{\mu}([w]_k) = \tilde{\mu}([w']_k)$.*

Proof: By lemma 4.5, $\tilde{\mu}([w]_k) = m^{-(n_1+n_2)}(\frac{1}{2})^{|w|}$. Since the number of paired α in w' is also n_1, we get that $\tilde{\mu}([w]_k) = \tilde{\mu}([w']_k)$. $\square$

Proposition 4.3 *If $w \equiv w'$(mod M), $|w| = |w'|$, and $k \in \mathbb{Z}$, then $\tilde{\mu}$ is $g_{w,w',k}$ invariant.*

Proof: First note that if $w \equiv w'(\text{mod } M)$ then for every $s, t \in L(X)$ $swt \equiv sw't(\text{mod } M)$. This fact, along with proposition 4.2 shows that $\tilde{\mu}(A) = \tilde{\mu}(g_{w,w',k}(A))$ for every cylinder set A. Since the cylinder sets generate the Borel sets, this shows $\tilde{\mu}$ is $g_{w,w',k}$-invariant. $\square$

For $x \in B_{-\infty}^{-\infty}$, and $j > 0$ define:

$$a_j(x) = \min\{k > 0 :\ H_{k+1}(x) = -j\}$$

$$b_j(x) = \max\{k < 0 :\ H_k(x) = -j\}$$

Note that for any $x \in B_{-\infty}^{-\infty}$, $(a_j(x))_{j\in\mathbb{N}}$ is strictly increasing, and $(b_j(x))_{j\in\mathbb{N}}$ is strictly decreasing. Also note that $x_{a_j(x)} \in \{\beta_1, \ldots, \beta_m\}$ and $x_{b_j(x)} \in \{\alpha_1, \ldots, \alpha_m\}$, and if $x_{a_j(x)} = \beta_i$ then $x_{b_j(x)} = \alpha_i$. Let $A_c^n = \{x \in B_{-\infty}^{-\infty} :\ x_{b_j(x)} = x_{b_{j+c}(x)}\ \forall j > n\}$.

Lemma 4.7 *$\tilde{\mu}(A_c^n) = 0$ for all $c \in \mathbb{Z} \setminus \{0\}$, $n \geq 0$*

Proof: For $z \in S_{-\infty}^{-\infty}$ define

$$\tilde{b}_j(z) = \max\{k < 0 : \ \tilde{H}_k(z) = j\}$$

For any $x \in B_{-\infty}^{-\infty}$, $\tilde{b}_j(z(x)) = b_j(x)$. Now, for $J \subset \mathbb{N}$ with $|J| < \infty$:

$$\tilde{\mu}(\{x_{b_j(x)} = x_{b_{j+c}(x)} \text{ for } j \in J \ \}) =$$

$$(\mu_2 \times \mu_1)(\{(a, z) : \ a_{l_{j,1}} = a_{l_{j,2}}, \ l_{j,1} = \tilde{b}_j(z) \ l_{j,2} = \tilde{b}_{j+c}(z) \text{ for } j \in J\}) = (\frac{1}{m})^{|J|}$$

This follows from the definition of $\tilde{\mu}$ as the image of a product measure, and from the fact that $(b_j(x))_{j \in \mathbb{N}}$ is strictly monotonic, so the $l_{j,1}$'s are all distinct, and $l_{j,1} \neq l_{j,2}$ for $j \in J$. Thus, $\tilde{\mu}(A_c^n) = 0$. □

Proposition 4.4 *There exists a double-tail invariant set $X_0 \subset X$ with $\tilde{\mu}(X_0) = 1$, such the countable set of $\mathcal{T}_2$-holonomies*

$$\mathcal{H} = \{g_{w,w',n} : \ n \in \mathbb{Z}, \ w, w' \in L(X) \ |w| = |w'|, \ w \equiv w' \, (mod \ M), \ \}$$

generates $\mathcal{T}_2(X_0)$.

Proof: Let $X_0 = B_{-\infty}^{-\infty} \setminus \bigcup_{n,m>0} \bigcup_{c \neq 0} T^{-m} A_c^n$. Since $\tilde{\mu}(B_{-\infty}^{-\infty}) = 1$, and $\tilde{\mu}(A_c^n) = 0$ for $c \neq 0$ by the previous lemma, $\tilde{\mu}(X_0) = 1$. Also, since $B_{-\infty}^{-\infty}$ and $\bigcup_{n,m>0} \bigcup_{c \neq 0} T^{-m} A_c^n$ are $\mathcal{T}_2$-invariant sets, X_0 is $\mathcal{T}_2$-invariant. We show that $\mathcal{H}$ generates $\mathcal{T}_2(X_0)$.

Suppose $(x, y) \in \mathcal{T}_2(X_0)$. We must show that $y = g(x)$ for some $g \in \mathcal{H}$. $\exists n \in \mathbb{N}$ so that $x_{[-n,n]^c} = y_{-[n,n]^c}$. Let $w = x_{[-n,n]}$, $w' = y_{[-n,n]}$. Let $c = H(w) - H(w')$. First assume $c \neq 0$. Let $x' = T^{-n}(x)$, $y' = T^{-n}(y)$. Then $x'_{[0,2n]^c} = y'_{[0,2n]^c}$. For all $k > 2n$, $H_k(x') = H_k(y') + c$. Therefore, $a_j(x') = a_{j+c}(y')$ for all $j > 2n + |c|$. Also, Since $x'_{[0,2n]^c} = y'_{[0,2n]^c}$, $H_k(x') = H_k(y')$ for all $k < 0$. So $b_j(x') = b_j(y')$ for all $j > 0$.

For $j > 2n + |c|$, denote $x'_{a_j(x')} = \beta_i$. Then $x'_{b_j(x')} = \alpha_i$. Also, $y'_{a_{j+c}(y')} = y'_{a_j(x')} = x'_{a_j(x')} = \beta_i$, so $y'_{b_{j+c}(y')} = \alpha_i$. Therefore, $x'_{b_{j+c}(x')} = y'_{b_{j+c}(y')} = \alpha_i$. We conclude that $x'_{b_j(x')} = x'_{b_{j+c}(x')}$ for all $j > 2n + |c|$. This proves that $x \in T^{-n} A_c^{2n+|c|}$, but we assumed $x \in X_0$, so this is a contradiction, so $c = 0$. Therefore, for every $k_1 < -n$ and $k_2 > n$, we have:

$$H_{k_1}(x) - H_{k_2}(x) = H_{k_1}(y) - H_{k_2}(y)$$

Let $N = \min\{k \geq n : \ H_{k+1}(x) < -2n\}$, and $N' = \max\{k < -n : \ H_k(x) = H_{N(x)+1}(x)\}$. N and N' are well defined for $x \in B_{-\infty}^{-\infty}$. We have that $H_{N+1}(x) - H_{N'}(x) = H_{N+1}(y) - H_{N'}(y) = 0$, and so $x_{[N',N]} \equiv y_{[N',N]} \equiv 0 \pmod{M}$. Thus $y = g_{x_{[N',N]}, y_{[N',N]}, N'}(x)$.
□

Proposition 4.5 *$\tilde{\mu}$ is a shift invariant probability.*

Proof: Let $[w]_k$ be a cylinder set.By lemma 4.5, we have:

$$\tilde{\mu}([w]_k) = m^{-n_1+n_2}(\frac{1}{2})^{|w|}$$

and also:

$$\tilde{\mu}(T^{-1}[w]_k) = m^{-n_1+n_2}(\frac{1}{2})^{|w|}$$

So $\tilde{\mu}(A) = \tilde{\mu}(T^{-1}[A])$ for every Borel set A. □

One could question whether proposition 4.5 follows immediately from the fact that the shift mapping is a normalizer of the double-tail relation. We note that in general double-tail invariant measures are not necessarily shift invariant. To see this, consider a (finite) subshift consisting of an orbit of a periodic point. For more elaborate examples of a similar phenomenon see [4], where it is shown that the "generalized hard core model" has Gibbs measures which are not shift-invariant.

Proposition 4.6

$$h_{\tilde{\mu}}(X,T) = \log(2) + \frac{1}{2}\log(m)$$

Proof: We have $h_{\tilde{\mu}}(X,T) = \lim_{n\to\infty} h_{\tilde{\mu}}(x_0|x_{-1}, x_{-2}, \ldots, x_{-n})$. Let

$$\varpi(a_1,\ldots,a_n) = \min\{H(a_1,\ldots,a_k) : \ 0 \le k \le n\}$$

By applying lemma 4.5, we get:

$$h_{\tilde{\mu}}(x_0|x_{-1} = a_1,\ldots,x_{-n} = a_n) = \begin{cases} \log(2m) & \text{if } \varpi(a_1,\ldots,a_n) \ge 0 \\ \log(2) + \frac{1}{2}\log(m) & \text{if } \varpi(a_1,\ldots,a_n) < 0 \end{cases}$$

We have $h_{\tilde{\mu}}(x_0|x_{-1}, x_{-2}, \ldots, x_{-n}) = \tilde{\mu}(\varpi(a_1,\ldots,a_n) < 0)(\log(2) + \frac{1}{2}\log(m)) + \tilde{\mu}(\varpi(a_1,\ldots,a_n) \ge 0)\log(2m)$. Since $\lim_{n\to\infty} \tilde{\mu}(\varpi(a_1,\ldots,a_n) \ge 0) = 0$, we have $h_{\tilde{\mu}}(X,T) = \log(2) + \frac{1}{2}\log(m)$. □

For $m \ge 2$, $h_{\tilde{\mu}}(X,T) < h_{top}(X,T)$. Thus, $\tilde{\mu}$ provides an example of a shift invariant probability, which is also $\mathcal{T}_2$ invariant, yet has entropy which is strictly less than the topological entropy, for $m \ge 2$.

4.3 No other Double-Tail Invariant Probabilities

In this subsection we conclude that apart from the two probabilities described in section 4.1 and the probability defined in section 4.2, there are no other ergodic double-tail invariant probabilities for the Dyck shift.

By lemma 4.2 we know that there are no more double-tail invariant probabilities on the sets $B^{+\infty}_{-\infty}$ and $B^{-\infty}_{+\infty}$. We also know by the same lemma that there are no such probabilities on $B^{\mathbb{R}}_{-\infty}$ and $B^{-\infty}_{\mathbb{R}}$.

Our next goal is to prove $\tilde{\mu}$ is unique on $B^{-\infty}_{-\infty}$:

Proposition 4.7 *Suppose ν is a $\mathcal{T}_2(B_{-\infty}^{-\infty})$ invariant probability. Then for every $w \equiv 1$ (mod M),*

$$\nu([w]_t) = (\frac{1}{2\sqrt{m}})^{|w|}$$

Proof: Let $[w]_t$ be a balanced cylinder with $|w| = 2n$. For $i < t$, Denote:

$$M_{i,i+2N} = \{x \in X : \; x_i^{i+2N} \equiv 1(\bmod M)\}$$

Since all balanced cylinders of the same length have equal ν- probability, we can calculate $\nu([w]_t \mid M_{i,i+2N})$ by counting the number of balanced words of length $2N$, and the number of such balanced words with w as a subword starting at position $t - i$. By lemma 3.2, the number of balanced words of length $2N$ is

$$w_{2N}^m = \frac{\binom{2N}{N}}{N+1} m^N$$

The number balanced word of length $2N$ with w as a subword starting at position $t - i$ is w_{2N-2n}^m. Thus,

$$\nu([w]_t \mid M_{i,i+2N}) = \frac{w_{2N-2n}^m}{w_{2N}^m}$$

It easily follows that:

$$\lim_{N\to\infty} \nu([w]_t \mid M_{i,i+2N}) = \lim_{N\to\infty} \frac{w_{2N}^m}{w_{2N-2n}^m} = (\frac{1}{2\sqrt{m}})^{2n}$$

Since $\nu(B_{-\infty}^{-\infty}) = 1$, we have

$$\nu(\bigcap_{N_0\in\mathbb{N}} \bigcup_{i\in-\mathbb{N}} \bigcup_{N>N_0} M_{i,i+2N}) = 1$$

For $N_0 > n$ define a random variable $\chi_{N_0}(x) := \min\{N > N_0 : \; x \in \bigcup_{i\in-\mathbb{N}} M_{i,i+2N}\}$. We have

$$\nu([w]_t) = \sum_{N>N_0} \nu(\chi_{N_0} = N)\nu([w]_t \mid \chi_{N_0} = N) \to (\frac{1}{2\sqrt{m}})^{2n}$$

□

Proposition 4.8 *$\tilde{\mu}$ is the unique $\mathcal{T}_2$ invariant probability on $B_{-\infty}^{-\infty}$.*

Proof: Suppose ν is a $\mathcal{T}_2$ invariant probability on $B_{-\infty}^{-\infty}$. By proposition 4.7,

$$\forall w \equiv 1(\bmod M) \; \nu([w]) = (\frac{1}{2\sqrt{m}})^{|w|} \tag{4}$$

For $a \in L(X)$, we say that $w \in L(X)$ is a *minimal balanced extension* of a, if the following conditions hold:

1. There exist $l, r \in L(X)$ such that $w = lar$.
2. $w \equiv 1 \pmod{M}$
3. For every l' suffix of l and r' prefix of r, $l'ar' \equiv 1$ implies $l'ar' = w$.

Since for every $a \in L(X)$,

$[a]_t =_\nu \biguplus\{[w]_s : w$ is a minimal balanced extension of a, with $(w_i)_{i=t-s}^{t-s+|w|} = a\}$

We have:

$$\nu([a]_t) = \sum_{[w]_s} \nu([w]_s) = \sum_{[w]_s} \tilde{\mu}([w]_s) = \tilde{\mu}([a]_t)$$

Where the sum ranges over minimal balanced extensions of a. This proves $\nu = \tilde{\mu}$. By theorem 4.1, this proves $\tilde{\mu}$ is the unique double tail invariant probability of $B_{-\infty}^{-\infty}$. □

Finally, we show that no other double-tail invariant probabilities exist for the Dyck Shift.

Define: $\hat{p} : \Sigma^{\mathbb{Z}} \mapsto \Sigma^{\mathbb{N}}$ by $\hat{p}((x_n)_{n\in\mathbb{Z}}) = (x_n)_{n\in\mathbb{N}}$. This is a Borel mapping that maps the two-sided Dyck shift X onto the one sided Dyck shift $Y \subset \Sigma^{\mathbb{N}}$.

Let $K_0 = \{x \in X : H_i(x) \geq 0, \forall i < 0\}$, and $K_i = T^{-i}(K_0))$. Notice that $B_t^s \subset \bigcup_{i=0}^{\infty} K_i$, for $s, t \in \{\{+\infty\}, \mathbb{R}\}$.

Lemma 4.8 *If $A, B \subset Y$ are Borel sets, and $g : A \mapsto B$ is a $\mathcal{T}(Y)$-holonomy, then there exists a $\mathcal{T}_2(X)$-holonomy $\tilde{g} : (\hat{p}^{-1}(A) \cap K_0) \mapsto (\hat{p}^{-1}(B) \cap K_0)$*

Proof: We define $\tilde{g} : (\hat{p}^{-1}(A) \cap K_0) \mapsto (\hat{p}^{-1}(B) \cap K_0)$ as follows:

$$\tilde{g}(x)_n = \begin{cases} x_n & n < 0 \\ g(\hat{p}(x))_n & n \geq 0 \end{cases}$$

We prove that $\tilde{g}$ takes $\hat{p}^{-1}(A) \cap K_0$ into $\hat{p}^{-1}(B) \cap K_0$. Let $x \in \hat{p}^{-1}(A) \cap K_0$. Since $x_n = \tilde{g}(x)_n$ for all $n < 0$, we have $H_n(x) = H_n(\tilde{g}(x))$ for $n < 0$. Because $x \in K_0$ we have $H_n(\tilde{g}(x)) \geq 0$ for $i < 0$. Let $y = \tilde{g}(x)$. Now we prove that $y \in X$. Otherwise, there exist $n_1, n_2 \in \mathbb{Z}$, such that $n_1 = \min\{l < n_2 : H_l(y) = H_{n_2+1}(y)\}$, and $y_{n_1} = \alpha_i$ $y_{n_2} = \beta_j$ with $i \neq j$. If $n_1, n2 < 0$ then $y_{n_1} = x_{n_1}$, $y_{n_2} = x_{n_2}$, so this contradicts the fact that $x \in X$. If $n_1, n2 \geq 0$, then $y_{n_1} = g(\hat{p}(x))_{n_1}$, $y_{n_2} = g(\hat{p}(x))_{n_2}$, so this contradicts the fact that $g(\hat{p}(x)) \in Y$. We remain with the case $n_1 < 0 \leq n_2$. We have $H_{n_1}(y) \geq 0 = H_0(y)$, and $H_{n_2+1}(y) = H_{n_2}(y) - 1$ (since $y_{n_2} = \beta_j$).Also, $H_{n_2+1}(y) = H_{n_1}(y) \geq 0$. Since $H_i(y) - H_{i+1}(y) = \pm 1$, there must be some $l > 0$ such that $H_l(y) = H_{n+1}(y)$. This contradicts the condition on $n_1, n2$. By the definition of $\tilde{g}$, $\hat{p}(\tilde{g}(x)) = g(\hat{p}(x))$, so $\tilde{g}(x) \in \hat{p}^{-1}(B)$. The fact that g is one to one and onto $(\hat{p}^{-1}(B) \cap K_0)$ follows from the fact that

$$\tilde{g}^{-1}(x)_n = \begin{cases} x_n & n < 0 \\ g^{-1}(\hat{p}(x))_n & n \geq 0 \end{cases}$$

To complete the proof of the lemma we must show that $(x, \tilde{g}(x)) \in \mathcal{T}_2(X)$. Since g is a $\mathcal{T}(Y)$-holonomy, $\hat{p}(x)$ and $g(\hat{p}(x)$ only differ in a finite number of (positive) coordinates. x and $\tilde{g}(x)$ only differ in the coordinates where $\hat{p}(x)$ and $g(\hat{p}(x))$ differ, which is a finite set. So $(x, \tilde{g}(x)) \in \mathcal{T}_2(X)$ $\square$

Lemma 4.9 *There are no $\mathcal{T}_2(X)$-invariant probability measures on X supported by B_t^s, $s, t \in \{\{+\infty\}, \mathbb{R}\}$.*

Proof: We first prove the result for $B_t^{\mathbb{R}}$,$t \in \{\{+\infty\}, \mathbb{R}\}$. Recall that $K_i = \{x \in X : H_n(x) \geq H_i(x), \forall n < i\}$. Notice that $B_t^{\mathbb{R}} \subset \bigcup_{i=0}^{\infty} K_i$.
Suppose μ is a $\mathcal{T}_2(X)$-invariant probability supported by $B_t^{\mathbb{R}}$,where $t \in \{\{+\infty\}, \mathbb{R}\}$, then $\mu(K_i) > 0$ for some $i \geq 0$. Without loss of generality we can assume $\mu(K_0) > 0$.
Define a probability $\breve{\mu}$ on Y by the formula:

$$\breve{\mu}(A) = \frac{\mu(\hat{p}^{-1}(A) \cap K_0)}{\mu K_0}$$

By lemma 4.8, $\breve{\mu}$ is a $\mathcal{T}(Y)$ invariant probability. Also, since $\mu(B_t^{\mathbb{R}}) = 1$,

$$\breve{\mu}(\{y \in Y : \liminf_{n \to +\infty} H_n(y) \in \mathbb{R}\}) = 1$$

Similarly, the existence of a $\mathcal{T}_2(X)$-invariant probability supported by $B_t^{+\infty}$,where $t \in \{\{+\infty\}, \mathbb{R}\}$ would result in a $\mathcal{T}(Y)$-invariant probability $\breve{\mu}$ with

$$\breve{\mu}(\{y \in Y : \liminf_{n \to +\infty} H_n(y) = +\infty\}) = 1$$

But in section 3 it was proved that the one sided Dyck shift has a unique $\mathcal{T}$-invariant probability, supported by

$$\{y \in Y : \liminf_{n \to +\infty} H_n(y) = -\infty\}$$

Which rules out the possibility that such $\breve{\mu}$ exists.

$\square$

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